JOURNAL OF DIFFERENTIAL EQUATIONS 61, 380–397 (1986)

Uniqueness of Non-negative Solutions of Semilinear Equations in $\mathbb{R}^n$

L. A. PELETIER

Department of Mathematics, University of Leiden, Leiden, The Netherlands

AND

JAMES SERRIN

Department of Mathematics, University of Minnesota, Minneapolis, Minnesota

Received March 12, 1984; revised January 7, 1985

In an earlier paper [6] we discussed the uniqueness of positive classical solutions $u(x)$ of the problem

$$(\mathrm{I})\qquad \begin{array}{lll} \Delta u + f(u) = 0 & \text{in} & \mathbb{R}^n \\ u(x) \to 0 & \text{as} & x \to \infty, \end{array}$$

in which $n > 1$ and $x = (x_1,..., x_n)$, under the following basic assumptions on the function f:

A1. f is defined and locally Lipschitz continuous on $(0, \infty)$.

A2. The integral $F(u) = \int_0^u f(s)\, ds$ exists for all $u > 0$ and is positive at some value $\delta > 0$.

By **A2** both of the quantities

$$\alpha = \inf\{u > 0 \colon f(u) > 0\}, \qquad \beta = \inf\{u > 0 \colon F(u) > 0\}$$

are well-defined (clearly $0 \leqslant \alpha \leqslant \beta$). It was shown that Problem (I) has at most one positive radial solution if f additionally satisfies the hypotheses:

H. $\lim_{u \to 0}(f(u)/u) = -m$ for some positive number m.

0022-0396/86 $3.00

S. The function

$$u \mapsto \frac{f(u)}{u-\beta}$$

is nonincreasing for all $u \in (\beta, \infty)$ where $f(u) > 0$.

Conditions **A1** and **A2** are quite mild, **A2** in fact being necessary for the existence of a solution. There are, however, no intrinsic reasons for assuming **H**, and it is the object of this paper to weaken this assumption considerably. We shall replace it by

H*. One of the following conditions holds:

(i) $n = 2$ and $\beta > 0$,

(ii) $n > 2$ and $\alpha > 0$.

Hypothesis **H** of course implies $\alpha > 0$, so that **H*** obviously is a considerably weaker condition than **H**. In particular, we note that **H*** allows the possibility $f \equiv 0$ for u near 0.

It is known that there can be no positive radial solution of (I) when

$$\int_0 \frac{du}{\sqrt{F(u)}} < \infty;$$

see [1] and Section 4. On the other hand, even though (I) cannot have positive radial solutions in this case, it *can* have solutions which are non-negative but not identically zero. Such behavior is interesting in itself; in order to include it in our considerations we shall henceforth allow not just the class of positive solutions but also those which are non-negative and non-trivial.

So that the equation be well-defined when $u = 0$ we naturally add the condition:

A3. $\lim_{u \to 0} f(u) = 0$ and $f(0) = 0$.

(In general, any limit other than 0 is incompatible with the condition $u(x) \to 0$ as $x \to \infty$.)

In work of Gidas, Ni, and Nirenberg [2, 3] it is shown that when **H** holds and $f \in C^{1+\varepsilon}$ near $u = 0$, then any positive solution of (I) is radially symmetric about some center of symmetry. In the generality of our present assumptions, a corresponding result is not yet known (and may be quite difficult to verify). At the same time, radially symmetric solutions of (I) are an important class of solutions in their own right, and in many cases they are he *only* possible global solutions. Consequently, it is reasonable to study the uniqueness of radially symmetric solutions for their own sake.

In the sequel we shall consider, then, the following radial problem associated with (I):

$$\text{(II)}\qquad \begin{aligned} &u''+((n-1)/r)\,u'+f(u)=0, && r>0, && (1)\\ &u\geqslant 0, && r\geqslant 0, && (1')\\ &u'(0)=0, \qquad \lim_{r\to\infty} u(r)=0, && && (1'') \end{aligned}$$

where u is a C^2 function of the radial variable r on $[0, \infty)$, with $u \not\equiv 0$. We shall assume throughout that f is a given function satisfying the natural conditions **A1**, **A2**, **A3**. For simplicity we shall not refer to these conditions in the statements of our results, it being tacitly understood, however, that they are always present.

Our purpose is to show that Problem (II) has at most one solution under the hypotheses **H*** and **S**. Thus we considerable generalize the results of [6] both by extending the class of solutions considered and by weakening the restrictions which are placed on f near $u=0$.

The general approach is as follows. We say that a solution of Problem (II) is of class $\mathscr{C}$ if

$$\lim_{r\to\infty} r^{n-1}u'(r) \text{ exists (finite).} \qquad (2)$$

This condition, which is crucial to our deliberations, is easily shown to follow from condition **H***(ii); see Lemma 5, part (i). We then prove

THEOREM 1. *Suppose $\beta>0$, and let u and v be two different solutions of Problem* (II). *Assume moreover either that $n=2$ or that both u and v are of class $\mathscr{C}$. Then the graphs of u and v must intersect at some point (R, U) with $U>0$.*

The proof of this theorem is modelled somewhat along the lines of the corresponding result in [6], but requires considerably more care because we can no longer rely on the strong hypothesis **H**. An interesting set of lemmas used in the proof concerns the *horizontal* separations of solution graphs (see Section 2). We subsequently obtain (see Section 3) a strengthened version of Theorem 1.

THEOREM 2. *Let the hypotheses of Theorem* 1 *hold. Then the graphs of u and v must intersect at some point (R, U) with $U>\beta$.*

The final step in the proof consists in showing that if condition **S** holds then the graphs of the solutions cannot intersect at any point (R, U) for which $U>\beta$. Together with Theorem 2 this yields the following conclusions.

THEOREM 3. *Let conditions* **H*** *and* **S** *hold. Then Problem* II *cannot have more than one solution.*

THEOREM 4. *Suppose* $\beta > 0$. *If condition* **S** *holds, there can be at most one solution of Problem* II *which is of class* $\mathscr{C}$.

The number n in Problem (II) denotes the number of space dimensions, and was assumed to be an integer greater than 1. If we drop this interpretation, and assume merely that n is a parameter in the range $(1, \infty)$, we find that Theorem 1 continues to hold while Theorems 2 and 3 require the extra condition $n \geqslant \frac{3}{2}$. For values of n in the range $(1, \frac{3}{2})$ the conclusion of Theorem 2 continues to hold however if we replace β by α. If we correspondingly strengthen condition **S** be replacing β by α this gives the following result.

Suppose $n \in (1, \frac{3}{2})$ *and let* **S** *hold with* β *replaced by* α. *Then Problem* (II) *cannot have more than one solution if* $\alpha > 0$.

Problems in which f fails to satisfy **H** occur frequently in applications. As an example we mention a model arising in population dynamics, due to Gurtin and MacCamy [4], describing the spread of biological populations. If u denotes the population density, one is led to the diffusion equation

$$u_t = \Delta(u^m) + u(1-u)(u-a) \qquad (m > 1, 0 < a < 1).$$

Let $\tilde{u}$ be an equilibrium solution of this equation. Since $\tilde{u}$ is a density, we may expect $\tilde{u} \geqslant 0$. Then $v = \tilde{u}^m$ satisfies Eq. (1) in which $f(v) \sim -av^{1/m}$ near $v = 0$, and hence, since $m > 1$, f satisfies condition **H*** but not **H**.

The outline of the paper is as follows. In Section 1 we give some general properties of solutions of Problem II, quoting from [6] when convenient. In Section 2 we consider various properties of solutions of class $\mathscr{C}$, including the main separation lemmas and Theorem 1. Section 3 contains the proofs of Theorems 2, 3, and 4.

Other uniqueness theorems for global solutions of Problems (I) and (II), which replace condition **S** by various alternate restrictions on f, have been obtained by K. McLeod and Serrin [5].

1. GENERAL BEHAVIOUR OF SOLUTIONS

We begin by deriving some basic identities.

LEMMA 1. *Let* $u = u(r)$ *be a solution of Eq.* (1) *on a finite interval* $[r_0, r_1] \subset \bar{\mathbb{R}}^+$ *and let* $u > 0$ *on* (r_0, r_1). *Then*

$$\left\{\frac{1}{2}u'(r_1)^2+F(u_1)\right\}-\left\{\frac{1}{2}u'(r_0)^2+F(u_0)\right\}=-(n-1)\int_{r_0}^{r_1}u'(r)^2\frac{dr}{r}, \quad (3)$$

where $u_i=u(r_i)$, $i=0, 1$.

Proof. Multiply Eq. (1) by u', and integrate over (r_0, r_1).

LEMMA 2. *Let u be a strictly monotone solution of Eq.* (1) *on a finite interval* $[r_0, r_1]\subset\bar{\mathbb{R}}^+$. *Let $r(u)$ be the inverse of $u(r)$. Then*

$$\text{(i)}\quad \tfrac{1}{2}r_1^{2n-2}u'(r_1)^2-\tfrac{1}{2}r_0^{2n-2}u'(r_0)^2=\int_{u_1}^{u_0}r^{2n-2}(u)\,f(u)\,du \quad (4)$$

$$\text{(ii)}\quad r_1^{2n-2}\{\tfrac{1}{2}u'(r_1)^2+F(u_1)\}-r_0^{2n-2}\{\tfrac{1}{2}u'(r_0)^2+F(u_0)\}$$

$$=2(n-1)\int_{r_0}^{r_1}r^{2n-3}F(u(r))\,dr, \quad (5)$$

where $u_i=u(r_i)$, $i=0, 1$.

Proof. Multiply Eq. (1) by r^{n-1}. Then we can write it as

$$(r^{n-1}u')'+r^{n-1}f(u)=0.$$

Next we multiply by $r^{n-1}u'$ and integrate over (r_0, r_1). There results

$$\tfrac{1}{2}r_1^{2n-2}u'(r_1)^2-\tfrac{1}{2}r_0^{2n-2}u'(r_0)^2=-\int_{r_0}^{r_1}r^{2n-2}f(u)u'\,dr. \quad (6)$$

Changing the integration variable to u in the integral yields (i). Since $f(u)u'=dF(u)/dr$, an integration by parts gives

$$\int_{r_0}^{r_1}r^{2n-2}f(u)u'\,dr=r^{2n-2}F(u(r))\Big|_{r_0}^{r_1}-2(n-1)\int_{r_0}^{r_1}r^{2n-3}F(u(r))\,dr. \quad (7)$$

Putting (6) and (7) together, we obtain (ii).

Next, we shall extend these identities to solutions of Problem (II). We first recall a lemma from [6].

LEMMA 3. *Let u be a solution of Problem* (II). *Then* $u'(r)\leqslant 0$ *for all* $r>0$. *Equality holds at a point $\bar{r}$ if and only if* $u(\bar{r})=0$.

Remark. Lemma 3 was proved in [6] only for positive solutions of (1). The conclusion there was stronger: $u'<0$. However, the argument given there can easily be generalized to non-negative solutions. Since $u'=0$ if $u=0$, the conclusion can only be $u'\leqslant 0$.

COROLLARY. *Let u be a solution of Problem* (II). *Then either $u>0$ on $[0,\infty)$, or $u>0$ on $[0,a)$ and $u=0$ on $[a,\infty)$ for some $a>0$.*

LEMMA 4. *Let u be a non-trivial solution of Problem* (II). *Then*

$$\begin{aligned} &\text{(i)} \quad \lim_{r\to\infty} u'(r)=0, \\ &\text{(ii)} \quad \frac{1}{2}u'(r^2)+F(u(r))=(n-1)\int_r^\infty u'(s)^2\frac{ds}{s}, \qquad r\geqslant 0. \end{aligned} \tag{8}$$

Proof. In view of the preceding Corollary, we may assume that $u>0$ on $[0,a)$ for some maximal $a>0$ which may be finite or infinite. Suppose $a<\infty$. Then (i) is obvious since $u'=0$ on $[a,\infty)$ and (ii) follows by setting $r_1=a$ in (3).

Next suppose $a=\infty$. We let $r_1\to\infty$ in (3). The right-hand side converges to some negative limit, or to $-\infty$. This implies that $\frac{1}{2}u'(r_1)^2+F(u(r_1))$ converges, though possibly to $-\infty$. However, as $r_1\to\infty$, $u(r_1)\to 0$ and hence $F(u(r_1))\to 0$. Thus $\frac{1}{2}u'(r_1)^2$ converges (clearly not to $-\infty$, and so, equally clearly) to some non-negative limit l^2. Since $u(r_1)\to 0$ as $r_1\to\infty$, we can only have $l=0$. This proves part (i) and moreover yields the relation

$$\lim_{r\to\infty}\{\tfrac{1}{2}u'(r)^2+F(u(r))\}=0. \tag{9}$$

Part (ii) now follows at once if we take $r_0\geqslant 0$ in (3), let $r_1\to\infty$, and then use (9).

Remark. Let u be a solution of Problem (II). If we set $r=0$ in Lemma 4(ii) we obtain

$$F(u(0))=(n-1)\int_0^\infty u'(r)^2\frac{dr}{r}>0.$$

Since $F\leqslant 0$ on $(0,\beta)$, this implies that $u(0)>\beta$.

LEMMA 5. *Let u be a solution of Problem* (II). *Then*

(i) *if $\alpha>0$, there exists a non-negative number L such that*

$$\lim_{r\to\infty} r^{n-1}u'(r)=-L \tag{10}$$

and

$$\lim_{r\to\infty} r^{2(n-1)}F(u(r))=0; \tag{11}$$

(ii) *if $\beta > 0$, there exists a non-negative number L_1 such that*

$$\lim_{r \to \infty} r^{2(n-1)}\{\tfrac{1}{2}u'(r)^2 + F(u(r))\} = \tfrac{1}{2}L_1^2 \tag{12}$$

and

$$r^{2(n-1)}\{\tfrac{1}{2}u'(r)^2 + F(u(r))\} = \tfrac{1}{2}L_1^2 - 2(n-1)\int_r^\infty t^{2n-3}F(u(t))\,dt. \tag{13}$$

If $n \leqslant 2$ then $L_1 = 0$.

Proof. If u vanishes for all sufficiently large r then (i) and (ii) follow immediately with $L = L_1 = 0$ (for (13), let $r_1 \to \infty$ in (5)). Thus let $u > 0$ on $[0, \infty)$.

(i) Since $u(r) \to 0$ as $r \to \infty$, and $f(u) \leqslant 0$ for $0 < u \leqslant \alpha$, we have $f(u(r)) \leqslant 0$ for r large. Hence, if we let $r_1 \to \infty$ in (4) the integral will eventually decrease monotonically. On the other hand, using (4) again, the integral is bounded below: for all $r_1 > r_0 \geqslant 0$

$$\int_{u_1}^{u_0} r^{2n-2}(u)\, f(u)\, du \geqslant -\tfrac{1}{2}r_0^{2(n-1)}u'(r_0)^2,$$

where $u_i = u(r_i)$, $i = 0, 1$. Hence the integral in (4) converges to some finite limit as $r_1 \to \infty$, from which (10) follows at once.

By l'Hôpital's rule

$$\lim_{r \to \infty} r^{2(n-1)}\left\{\frac{1}{2}u'(r)^2 + F(u(r))\right\} = \lim_{r \to \infty} \frac{\frac{1}{2}u'(r)^2 + F(u(r))}{r^{-2(n-1)}}$$

$$= \lim_{r \to \infty} \frac{u'u'' + f(u)u'}{-2(n-1)\,r^{-2n+1}}$$

$$= \frac{1}{2}\lim_{r \to \infty} r^{2(n-1)}u'(r)^2 \qquad \text{(using (1))}$$

$$= \frac{1}{2}L^2 \qquad \text{(by (10))}.$$

Thus, again in view of (10),

$$\lim_{r \to \infty} r^{2(n-1)}F(u(r)) = 0.$$

(ii) The proof of (12) is almost identical to that of (10), except that we use the identity (5). To prove that the integral is bounded below as $r_1 \to \infty$, we now use (8) to obtain $\frac{1}{2}u'(r)^2 + F(u(r)) > 0$ for $r \geqslant 0$.

To show that $L_1 = 0$ when $n \leqslant 2$, suppose for contradiction that L_1 were positive in this case. Then by (12), since $F \leqslant 0$ for u near 0, we have

$$-r^{n-1}u' \geqslant \tfrac{1}{8}L_1 > 0$$

for all sufficiently large r. A simple integration then shows that $u \to \infty$ as $r \to \infty$, an obvious contradiction. The proof is therefore completed.

Remark. If $\alpha = 0$ but $\beta > 0$, Lemma 5 does not give an estimate for the behavior of either $u(r)$ or $u'(r)$ as $r \to \infty$. On the other hand, if we assume that u is of class $\mathscr{C}$, with

$$\lim_{r \to \infty} r^{n-1}u'(r) = -L \tag{14}$$

then $L_1 = L$ and

$$\lim_{r \to \infty} r^{2(n-1)}F(u(r)) = 0 \tag{15}$$

as can be seen from an argument similar to the one used to prove (11).

2. Separation of Solution Graphs

Let $u(r)$ and $v(r)$ be solutions of Problem (II), and let $u(r) \not\equiv v(r)$. By Lemma 3 and its corollary $u'(r) < 0$ and $v'(r) < 0$ when $u(r) > 0$ and $v(r) > 0$. Hence their inverses $r(u)$ and $s(u)$ are respectively well-defined for u in $(0, u(0))$ and u in $(0, v(0))$. Note that $u(0) \neq v(0)$, for otherwise the two solutions would be identical.

We begin with two preliminary lemmas.

Lemma 6. *The function $r(u) - s(u)$ can have at most a finite number of zeros if one of the following conditions is satisfied*:

(i) $\alpha > 0$, $n > 1$;

(ii) $\beta > 0$, $n \geqslant \frac{3}{2}$.

Proof. The proof of part (i) is essentially the same as that in [6], so we omit it.

To prove part (ii), we observe, as in [6], that the points of intersection are isolated, and thus, that if there exists an infinite number of them, they can be enumerated, say $u_1 > u_2 > u_3 \ldots$, and $u_k \to 0$ as $k \to \infty$. Let η_0 be a particular zero with $\eta_0 \leqslant \beta$ and set $r_0 = r(\eta_0)$. We can assume without loss of generality that $r > s$ in some right neighborhood of $u = \eta_0$.

If there exists a zero of $r - s$ on $(0, \eta_0)$, there must exist a number $\eta_1 \in (0, \eta_0)$ such that

$$r > s,\ r' < s' \text{ on } (\eta_1, \eta_0) \qquad \text{and} \qquad r'(\eta_1) = s'(\eta_1). \tag{16}$$

Set $r_1 = r(\eta_1)$ and $s_1 = s(\eta_1)$. If we apply the identity (5) for u on (r_0, r_1) and for v on (r_0, s_1) and then subtract we obtain

$$(r_1^{2n-2} - s_1^{2n-2})\{\tfrac{1}{2}u'(r_1)^2 + F(\eta_1)\} - \tfrac{1}{2}r_0^{2n-2}\{u'(r_0)^2 - v'(r_0)^2\}$$
$$= -\int_{\eta_1}^{\eta_0} \{r(u)^{2n-2} - s(u)^{2n-2}\}' F(u)\, du. \tag{17}$$

Since $0 > u'(r_0) > v'(r_0)$ we find, in view of Lemma 4(ii), that the left-hand side of (17) is positive. For $u \in (\eta_1, \eta_0)$ we have

$$\{r(u)^{2n-2} - s(u)^{2n-2}\}' = (2n-2)\{r^{2n-3}(r-s)' + (r^{2n-3} - s^{2n-3})s'\} < 0,$$

by (16) and the fact that $n \geqslant \frac{3}{2}$. Remembering that $(\eta_1, \eta_0) \subset (0, \beta)$, and hence that $F(u) \leqslant 0$ in (17), it follows that the right-hand side of (17) is non-positive.

Thus, there cannot be a zero of $r - s$ on $(0, \eta_0)$, and hence the number of zeros of $r - s$ must be finite.

Remark. The proof of Lemma 6(ii) yields in addition our first result on the horizontal separation of solution graphs.

Let u and v be solutions of Eq. (1) which intersect at a point (r_0, η_0). *If* $\beta > 0$, $\eta_0 \in (0, \beta]$, *and* $n \geqslant \frac{3}{2}$ *we have*

$$r'(\eta_0) < s'(\eta_0) \textit{ implies } r' < s' \textit{ on } (0, \eta_0], \tag{18}$$

If $1 < n < \frac{3}{2}$, *then* (18) *still holds if* $\alpha > 0$ *and* $\eta_0 \in (0, \alpha]$.

The conclusion for $1 < n < \frac{3}{2}$ follows when we use (4) instead of (5) in the proof of Lemma 6.

In the next set of lemmas we further consider the function $r(u) - s(u)$. Since the difference $r - s$ can have at most a finite number of zeros necessarily either $r - s > 0$ or $r - s < 0$ for all u sufficiently small. Without loss of generality we can always suppose the former to occur.

LEMMA 7. *Suppose* $r(u) - s(u) > 0$ *one some interval I. Then* $r(u) - s(u)$ *can have at most one critical point on I. Moreover, this critical point is a strict maximum.*

Since the proof of Lemma 7 in [6] is slightly more complicated than necessary, we give here a simpler version.

Proof of Lemma 7. We have

$$u_r = 1/r_u, \qquad u_{rr} = -r_{uu}/r_u^3.$$

Hence $r(u)$ satisfies the equation

$$r_{uu} - \frac{n-1}{r} r_u^2 - f(u) r_u^3 = 0.$$

A similar equation can be derived for $s(u)$. At a critical point we have $r > s$ and $r_u = s_u < 0$, so that by subtraction

$$(r-s)_{uu} = (n-1)\left(\frac{1}{r} - \frac{1}{s}\right) r_u^2 < 0.$$

Thus $r - s$ can only have a strict maximum.

We next consider solutions u and v satisfying condition $\mathscr{C}$ in the Introduction, with $n > 2$. We write

$$\lim_{r \to \infty} r^{n-1} u'(r) = -L, \qquad \lim_{r \to \infty} r^{n-1} v'(r) = -M.$$

LEMMA 8. *Suppose u and v satisfy condition $\mathscr{C}$, with $n > 2$. If $r(u) > s(u)$ for $0 < u < u_0$, then $L \geqslant M$.*

Proof. Suppose to the contrary that $L < M$. By l'Hôpital's rule

$$\frac{u(r)}{r^{2-n}} \to \frac{L}{n-2}, \quad \frac{v(r)}{r^{2-n}} \to \frac{M}{n-2} \qquad \text{as} \quad r \to \infty.$$

Hence, if we subtract, there results

$$r^{n-2}\{u(r) - v(r)\} \to \frac{L-M}{n-2} < 0$$

which contradicts the assumption that $r > s$ in $(0, u_0)$.

LEMMA 9. *Suppose u and v satisfy condition $\mathscr{C}$, with $n > 2$. Then if $\beta > 0$ and $r > s$ for $0 < u < u_0$ we have*

$$\{r(u) - s(u)\}' < 0 \qquad \textit{for} \quad 0 < u < u_0.$$

Proof. By Lemma 8 we have $L \geqslant M$. We now distinguish two cases: (i) $L > M$ and (ii) $L = M$.

(i) $L > M$. Choose $\varepsilon = \frac{1}{3}(L-M)$. Then we find, as in the proof of Lemma 8, that for r sufficiently large

$$u(r) > \frac{L-\varepsilon}{n-2} r^{-(n-2)} = u^*(r)$$

$$v(r) < \frac{M+\varepsilon}{n-2} r^{-(n-2)} = v^*(r).$$

Let r^* and s^* be the respective inverses of u^* and v^*. Then for u sufficiently small

$$r(u) - s(u) > r^*(u) - s^*(u) = \left\{ \left(\frac{L-\varepsilon}{n-2} \right)^{1/(n-2)} - \left(\frac{M+\varepsilon}{n-2} \right)^{1/(n-2)} \right\} u^{-1/(n-2)}.$$

Since $L - \varepsilon > M + \varepsilon$ we conclude that $r(u) - s(u) \to \infty$ as $u \to 0$, whence, by Lemmma 7, $\{r(u) - s(u)\}' < 0$ for $u \in (0, u_0)$.

(ii) $L = M$. By Lemma 4

$$\frac{1}{2} u'(r(u))^2 + F(u) = -(n-1) \int_0^u \frac{u'(r(t))}{r(t)} \, dt \tag{19}$$

and similarly for v, with r replaced by s. If we subtract the identity for v from that for u and write

$$a(u) = |u'(r(u))| \qquad \text{and} \qquad b(u) = |v'(s(u))|$$

we obtain (recall that $u', v' < 0$)

$$\begin{aligned} a(u)^2 - b(u)^2 &= 2(n-1) \int_0^u \left\{ \frac{a(t)}{r(t)} - \frac{b(t)}{s(t)} \right\} dt \\ &< 2(n-1) \int_0^u \frac{a(t) - b(t)}{r(t)} \, dt \end{aligned} \tag{20}$$

because $s < r$ on $(0, u_0)$.

Define

$$w(u) = \int_0^u \frac{a-b}{r} \, dt,$$

so that (20) becomes, with $' = d/du$,

$$(a+b)\, r w' < 2(n-1) w.$$

Integrating this inequality leads to

$$w(u) < w(\bar{u}) \exp\left\{2(n-1) \int_{\bar{u}}^{u} \frac{dt}{(a+b)\,r}\right\} \tag{21}$$

for $0 < \bar{u} < u < u_0$.

Assertion. $w(u) < 0$ for $0 < u < u_0$.

Assuming the assertion for the moment, we can readily complete the proof. Indeed if $w < 0$, then $a < b$ by (20), that is,

$$|u'(r(u))| < |v'(s(u))|, \qquad 0 < u < u_0,$$

or, in turn $\{r(u) - s(u)\}' < 0$ for $0 < u < u_0$, which was to be proved.

By Lemma 7 either $r(u) - s(u)$ is everywhere decreasing on $0 < u < u_0$ or $r(u) - s(u)$ is increasing for u near 0. In the first case $a(u) - b(u) < 0$ on $0 < u < u_0$, and in the second $a(u) - b(u) > 0$ for u near 0. Hence, by the definition of $w(u)$, either $w < 0$ for $0 < u < u_0$, and we are done, or $w(u) > 0$ for all u sufficiently small, say $0 < u < \delta$. Suppose then for contradiction that the latter case holds. Choose $\bar{u} \in (0, \delta)$. Then $w(\bar{u}) > 0$ and, by (21),

$$\begin{aligned} w(u) &< w(\bar{u}) \exp\left\{2(n-1) \int_{\bar{u}}^{u} \frac{dt}{ar}\right\} \\ &= w(\bar{u}) \exp\left\{2(n-1) \int_{\bar{r}}^{r} \frac{u'\,dr}{-u'r}\right\}, \end{aligned}$$

where $\bar{r} = r(\bar{u})$ and $0 < \bar{u} < u < u_0$. Thus $w(u) < w(\bar{u}) \exp\{2(n-1) \log(\bar{r}/r)\}$, or

$$r^{2(n-1)} w(u) < \bar{r}^{2(n-1)} w(\bar{u}), \qquad 0 < \bar{u} < u < u_0. \tag{22}$$

Since $b > 0$ we have from the definition of $w(u)$, for any $u > 0$,

$$\begin{aligned} r^{2(n-1)} w(u) &< r^{2(n-1)} \int_0^u \frac{a}{r}\,du \\ &= r^{2(n-1)} \int_r^{\infty} \frac{u'^2}{r}\,dr \\ &= \frac{r^{2(n-1)}}{n-1} \left\{\frac{1}{2} u'(r)^2 + F(u(r))\right\} \end{aligned}$$

by Lemma 4. Now from the remark after Lemma 5 (here we use the condition $\beta > 0$ and the fact that u is of type $\mathscr{C}$) there results

$$\limsup_{u \to 0} r^{2(n-1)} w(u) \leqslant \frac{L^2}{2(n-1)}. \tag{23}$$

Next, let $\bar{u} \to 0$ in (22) and use relation (23) (with the dummy variable u replaced by $\bar{u}$). This give

$$r^{2(n-1)} w(u) \leqslant \limsup_{\bar{u} \to 0} \bar{r}^{2(n-1)} w(\bar{u}) \leqslant \frac{L^2}{2(n-1)}, \qquad 0 < u < u_0.$$

If $L = 0$ this implies $w(u) \leqslant 0$ on $(0, u_0)$, contradicting the assumption that $w(u) > 0$ on $(0, \delta)$. Thus $w(u) < 0$ on $(0, u_0)$ in this case.

If $L > 0$, we need a more delicate estimate of the behavior of $r^{2(n-1)} w(u)$ as $u \to 0$. By l'Hôpital's rule we have (recall that $w(u) \to 0$ as $u \to 0$)

$$\begin{aligned}
\lim_{u \to 0} r^{2(n-1)} w(u) &= \lim_{u \to 0} \frac{\int_0^u ((a-b)/r)\, dt}{r^{-2(n-1)}} \\
&= \lim_{u \to 0} \frac{(a-b) u'}{-2(n-1)\, r^{-2(n-1)}} \\
&= -\frac{1}{2(n-1)} \lim_{r \to \infty} (r^{n-1} u')^2 \lim_{u \to 0} \left\{ \frac{v'(s(u))}{u'(r(u))} - 1 \right\} \\
&= -\frac{L^2}{2(n-1)} \left\{ \lim_{u \to 0} \frac{v'}{u'} - 1 \right\}
\end{aligned}$$

assuming the last limit to exist.

Because $r^{n-1} u'(r) \to -L$ and $r^{n-2} u(r) \to L/(n-2)$ (see Lemma 8), and since $L > 0$, we see that

$$\lim_{u \to 0} \frac{u'(r(u))}{u^{(n-1)/(n-2)}} = \lim_{r \to \infty} \frac{r^{n-1} u'(r)}{(r^{n-2} u(r))^{(n-1)/(n-2)}} = -L \left(\frac{L}{n-2} \right)^{-(n-1)/(n-2)}.$$

Similarly, since $L = M > 0$,

$$\lim_{u \to 0} \frac{v'(s(u))}{u^{(n-1)/(n-2)}} = -L \left(\frac{L}{n-2} \right)^{-(n-1)/(n-2)}.$$

Consequently

$$\lim_{u \to 0} \frac{v'(s(u))}{u'(r(u))} = 1$$

and so we find

$$\lim_{u \to 0} r^{2(n-1)} w(u) = 0.$$

The rest of the proof is the same as in the case $L = 0$.

Lemma 9 is crucial for the proof of Theorem 1 when u and v are of class $\mathscr{C}$. The corresponding result when $n \leqslant 2$ is the following

LEMMA 10. *Suppose $r > s$ for $0 < u < u_0$. Then if $\beta > 0$ and $n \leqslant 2$ we have*

$$\{r(u) - s(u)\}' < 0 \qquad \textit{for} \quad 0 < u < u_0.$$

Proof. By Lemma 5(ii) we have

$$r^{2(n-1)}\{\tfrac{1}{2}|u'|^2 + F(u)\} \to 0 \qquad \text{as} \quad r \to \infty.$$

The result now follows exactly as in the proof of Lemma 9, part (ii), since the condition (23) yields $\lim \bar{r}^{2(n-1)} w(\bar{u}) = 0$. Note in particular that the final step in the proof of Lemma 9 (the possibility $L > 0$) is not needed here.

The proof of Theorem 1 now follows as a simple consequence of Lemmas 9 and 10. Indeed suppose u and v are solutions of Problem (II) of class $\mathscr{C}$ which do not intersect at any positive value of u. Without loss of generality we may assume that $u > v$ as long as $u > 0$, and hence, that $r(u) - s(u) > 0$ on $(0, v(0)]$. By Lemma 9,

$$(r - s)' < 0 \qquad \text{for} \quad u \in (0, v(0)).$$

On the other hand,

$$(r - s)' \to \infty \qquad \text{as} \quad u \to v(0),$$

because $s'(u) \to -\infty$ and $r'(u)$ remains bounded as $u \to v(0)$. This contradiction shows that the graphs of u and v must intersect at some positive u. For $n \leqslant 2$ the proof is the same except that we use Lemma 10 instead of Lemma 9.

3. PROOFS OF THEOREMS 2, 3 AND 4

Let u and v be two different solutions of Problem (II), both of class $\mathscr{C}$, and let their last point of intersection (for $u > 0$) be (R, U). By Lemma 6 and Theorem 1 this point exists. By (ii) the Remark following Lemma 5 we have

$$R^{2n-2}\{\tfrac{1}{2}u'(R)^2+F(U)\}=\tfrac{1}{2}L^2-2(n-1)\int_R^\infty t^{2n-3}F(u(t))\,dt;$$

of course the same identity holds for v, provided L is replaced by M. Thus we obtain upon subtraction

$$\tfrac{1}{2}R^{2n-2}\{v'(R)^2-u'(R)^2\}+\tfrac{1}{2}(L^2-M^2)=-\int_0^U \Psi'(u)\,F(u)\,du, \qquad (24)$$

where $\Psi(u)=r^{2n-2}-s^{2n-2}$.

Suppose, without loss of generality, that $r>s$ on $(0, U)$. Then $|v'(R)|>|u'(R)|$ and, by Lemma 8, $L\geqslant M$. Hence the left-hand side of (24) is positive.

From Lemma 9 we deduce that

$$\Psi'=2(n-1)\{(r^{2n-3}-s^{2n-3})\,r'+s^{2n-3}(r-s)'\}<0 \qquad \text{on} \quad (0, U)$$

for $n\geqslant\frac{3}{2}$. Therefore, if $U\leqslant\beta$, then $F(u)\leqslant 0$ on the whole of $(0, U)$ and the right-hand side of (24) is non-positive. Remembering that the left-hand side is positive, we conclude that $U>\beta$. This completes the proof of Theorem 2 for solutions of class $\mathscr{C}$. When $n\leqslant 2$ the argument is essentially the same except that $L_1=M_1=0$ by Lemma 5(ii), and Lemma 10 must be used at the final stage instead of Lemma 9.

The proofs of Theorems 3 and 4 are now completed as in [6]. One shows that the graph of two solutions cannot intersect above the line $u=\beta$ if **S** is satisfied [6, Lemma 10]. Since, by Theorem 2, the graphs of two solutions must necessarily intersect above the line $u=\beta$, the proof is complete.

Finally, let us consider the case $n\in(1, \frac{3}{2})$. If we let $r_1\to\infty$ in (4) and set $r_0=R$, we obtain from Lemma 5(i)

$$\tfrac{1}{2}R^{2n-2}u'(R)^2-\tfrac{1}{2}L^2=-\int_0^U r(u)^{2n-2}f(u)\,du.$$

For v one obtains a similar expression with L replaced by M. Subtraction yields

$$\tfrac{1}{2}R^{2n-2}\{v'(R)^2-u'(R)^2\}+\tfrac{1}{2}(L^2-M^2)=\int_0^U \Psi(u)\,f(u)\,du.$$

Assuming again that $r>s$ on $(0, U)$ we see that the left-hand side is positive and that the right-hand side is negative if $U\leqslant\alpha$. Thus $U>\alpha$. The argument for the case $n\in(1, \frac{3}{2})$ is then completed as before, with β in **S** replaced by α, and $\alpha>0$.

4. Compact Support

We denote by $u(r)$ a decreasing solution of Eq. (1) in $[R, \infty)$ with $U = u(R) > 0$ and $u(r) \to 0$ as $r \to \infty$. Let $r(u)$ denote as usual the inverse of $u(r)$.

Lemma 11. *If $U \in (0, \beta)$ then*

$$r(u) \leqslant r(U) + 2^{-1/2} \int_u^U |F(s)|^{-1/2}\, ds, \qquad 0 < u \leqslant U.[1] \tag{25}$$

Proof. By Lemma 4

$$\tfrac{1}{2} u'(r)^2 + F(u(r)) \geqslant 0 \qquad \text{for} \quad r \geqslant R$$

or, because $u(r) \in [0, \beta)$ and hence $F(u(r)) \leqslant 0$,

$$\tfrac{1}{2} u'(r)^2 \geqslant |F(u(r))| \qquad \text{for} \quad r \geqslant R.$$

Since $u' \leqslant 0$ this implies that

$$u'(r) \leqslant -2^{1/2} |F(u(r))|^{1/2}. \tag{26}$$

By the corollary to Lemma 3 there exists a number $a \leqslant \infty$ such that $u(r) > 0$ when $r < a$ and $u(r) = 0$ when $r \geqslant a$. Thus $R < a$, and we obtain from (26)

$$|F(u(r))|^{-1/2} u'(r) \leqslant -2^{1/2} \qquad \text{for} \quad R \leqslant r < a.$$

Integrating this inequality from R to $r \in (R, a)$ we obtain

$$-\int_u^U |F(s)|^{-1/2}\, ds \leqslant -2^{1/2} [r(u) - R],$$

or

$$r(u) \leqslant R + 2^{-1/2} \int_u^U |F(s)|^{-1/2}\, ds.$$

Theorem 5. *Let u be a solution of Problem* II. *The condition*

$$\int_0 |F(s)|^{-1/2}\, ds < \infty \tag{27}$$

[1] If the integral fails to exist, then (25) holds vacuously.

is sufficient for u *to have compact support. If* $\alpha > 0$, *this condition is also necessary.*

Proof. That the condition (27) is sufficient for u to have compact support is an immediate consequence of (25).

To prove that (27) is necessary we assume that u has compact support $[0, a]$ with $a < \infty$.

By Lemma 4 we have

$$u'(r)^2 \leqslant 2\,|F(u)(r))| + c \int_r^a u'(s)^2\,ds, \qquad a-\delta < r < a,$$

where $c \equiv 2(n-1)/(a-\delta)$ and $\delta \in (0, a)$ will be chosen later. Applying Gronwall's lemma we deduce that

$$u'(r)^2 \leqslant 2\,|F(u(r))| + 2c \int_r^a |F(u(s))|\, e^{c(s-r)}\,ds. \tag{28}$$

Now we choose δ so small that $u(r) \in (0, \alpha)$ for $a-\delta < r < a$. Then

$$\frac{d}{dr}\,|F(u(r))| = -f(u(r))\,u'(r) \leqslant 0$$

and hence $|F(u(s))| \leqslant |F(u(r))|$. Using this in (28) we obtain

$$u'(r)^2 \leqslant 2\,|F(u(r))| \left\{1 + c \int_r^a e^{c(s-r)}\,ds\right\} \leqslant 2\,|F(u(r))|\, e^{c\delta}. \tag{29}$$

To complete the proof we proceed as in the proof of Lemma 11. Dividing by $|F|$ in (29) and taking the square root, we obtain

$$-\,|F(u(r))|^{-1/2}\, u'(r) \leqslant 2^{1/2} e^{\delta(n-1)/(a-\delta)}.$$

This yields, upon integration over $(a-\delta, a-\varepsilon)$, $0 < \varepsilon < \delta$,

$$\int_{u(a-\varepsilon)}^{u(a-\delta)} |F(u)|^{-1/2}\,du \leqslant 2^{1/2}\delta e^{\delta(n-1)/(a-\delta)}.$$

Because $u(a-\varepsilon) \to 0$ as $\varepsilon \to 0$, we conclude that

$$\int_0 |F(u)|^{-1/2}\,du < \infty,$$

which was to be proved.

ACKNOWLEDGMENT

This work was supported by a grant from the Netherlands Organization for the Advancement of Pure Research (ZWO) and by a grant from the National Science Foundation.

REFERENCES

1. PH. BENILAN, H. BREZIS, AND M. G. CRANDALL, A semilinear elliptic equation in $L^1(\mathbb{R}^N)$, *Ann. Scuola Norm Sup. Pisa* **2** (1975), 523–555.
2. B. GIDAS, W.-M. NI, AND L. NIRENBERG, Symmetry and related properties via the maximum principle, *Comm. Math. Phys.* **68** (1979), 209–243.
3. B. GIDAS, W.-M. NI, AND L. NIRENBERG, Symmetry of positive solutions nonlinear elliptic equations in $\mathbb{R}^n$, *Math. Anal. Appl. Adv. in Math. Suppl. Stud.* **7A** (1981), 369–402.
4. M. E. GURTIN AND R. C. MACCAMY, On the diffusion of biological populations, *Math. Biosci.* **33** (1977), 35–49.
5. K. MCLEOD AND J. SERRIN, Uniqueness of the ground state solution for $\Delta u + f(u) = 0$, *Proc. Nat. Acad. Sci. USA* **78** (1981), 6592–6595.
6. L. A. PELETIER AND J. SERRIN, Uniqueness of positive solutions of semilinear equations in $\mathbb{R}^n$, *Arch. Rational Mech. Anal.* **81** (1983), 181–197.